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Completeness of security markets and solvability of linear backward stochastic differential equations ☆

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Abstract

For a standard Black–Scholes type security market, completeness is equivalent to the solvability of a linear backward stochastic differential equation (BSDE, for short). An ideal case is that the interest rate is bounded, there exists a bounded risk premium process, and the volatility matrix has certain surjectivity. In this case the corresponding BSDE has bounded coefficients and it is solvable leading to the completeness of the market. However, in general, the risk premium process and/or the interest rate could be unbounded. Then the corresponding BSDE will have unbounded coefficients. For this case, do we still have completeness of the market? The purpose of this paper is to discuss the solvability of BSDEs with possibly unbounded coefficients, which will result in the completeness of the corresponding market.

Keywords: Completeness of market; Backward stochastic differential equations; Exponential process

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1. Introduction

Consider a security market of Black–Scholes type [6,12]: There are $(n+1)$ assets continuously traded in the market. The 0th asset is a bond, and the last n are stocks. The price process of the ith asset is denoted by $P_i(\cdot)$ and the following system of stochastic differential equations (SDEs, for short) is satisfied by $P_i(\cdot)$'s:

$$\begin{cases} dP_0(t) = r(t)P_0(t)\,dt, \\ dP_i(t) = b_i(t)P_i(t)\,dt + P_i(t)\langle \sigma_i(t), dW(t)\rangle, \quad 1 \leqslant i \leqslant n, \\ P_i(0) = p_i, \quad 0 \leqslant i \leqslant n, \end{cases} \tag{1.1}$$

where $W(\cdot)$ is a d-dimensional standard Brownian motion defined on some complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbf{P})$, $r(\cdot)$, $b_i(\cdot)$ and $\sigma_i(\cdot) \equiv (\sigma_{i1}(\cdot), \dots, \sigma_{id}(\cdot))^T$ are called the *interest rate* (of the bond), the *appreciation rate*, and the *volatility* (of the stocks), respectively. We denote $b(\cdot) = (b_1(\cdot), \dots, b_n(\cdot))^T$ and $\sigma(\cdot) = (\sigma_{ij}(\cdot))_{n\times d}$. Throughout this paper, we assume that processes $r(\cdot)$, $b(\cdot)$, and $\sigma(\cdot)$ are $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted. For the time being, we do not assume any integrability on these processes.

In the market (1.1), we consider a European contingent claim whose payoff at $t = T$ is given by an $\mathcal{F}_T$-measurable random variable ξ (we will identify ξ with such a contingent claim below). It is standard by now that replication of such a contingent claim amounts to solving the following backward stochastic differential equation (BSDE, for short) (see [1,3,8–11,16–18], for relevant results on BSDEs):

$$\begin{cases} dY(t) = \{r(t)Y(t) + \langle b(t) - r(t)\mathbf{1}, \pi(t)\rangle\}\,dt \\ \qquad\qquad + \langle \pi(t), \sigma(t)\,dW(t)\rangle, \quad t \in [0, T], \\ Y(T) = \xi, \end{cases} \tag{1.2}$$

where $\mathbf{1} \triangleq (1, \dots, 1)^T \in \mathbb{R}^n$, $\pi(\cdot)$ is a portfolio process (whose ith component $\pi_i(\cdot)$ represents the market value of the ith asset held by the investor). Any $\{\mathcal{F}\}_{t\geqslant 0}$-adapted process $(Y(\cdot), \pi(\cdot))$ taking values in $\mathbb{R} \times \mathbb{R}^n$ satisfying (1.2) in the usual Itô sense is called an adapted solution to BSDE (1.2). Thus, a contingent claim ξ is replicatable on $[0, T]$ if one can find an adapted solution $(Y(\cdot), \pi(\cdot))$ to (1.2). In this case, we usually refer to $Y(\cdot)$ and $\pi(\cdot)$ as replicating price process and replicating strategy process, respectively. By a rough definition, if for any ξ, BSDE (1.2) admits an adapted solution $(Y(\cdot), \pi(\cdot))$, the market is said to be complete. Thus, the completeness of the market is equivalent to the solvability of BSDE (1.2).

Now, suppose

$$\operatorname{rank} \sigma(t) = d, \quad \text{a.e. } t \in [0, T], \text{ a.s.}, \tag{1.3}$$

which implies that $n \geqslant d$ and $[\sigma(t)^T\sigma(t)]^{-1}$ exists. Let

$$\theta(t) = \big[\sigma(t)^T\sigma(t)\big]^{-1}\sigma(t)^T\big[b(t) - r(t)\mathbf{1}\big], \quad t \in [0, T], \tag{1.4}$$

and consider the following BSDE:

$$\begin{cases} dY(t) = \{r(t)Y(t) + \langle \theta(t), Z(t)\rangle\}\,dt + \langle Z(t), dW(t)\rangle, \quad t \in [0, T], \\ Y(T) = \xi. \end{cases} \tag{1.5}$$

Suppose BSDE (1.5) admits an adapted solution $(Y(\cdot), Z(\cdot))$. Define

$$\pi(t) = \sigma(t)\big[\sigma(t)^T\sigma(t)\big]^{-1}Z(t), \quad t \in [0,T]. \tag{1.6}$$

Then (noting (1.4))

$$\begin{cases} \sigma(t)^T\pi(t) = Z(t), \\ \langle\theta(t), Z(t)\rangle = \langle b(t) - r(t)\mathbf{1}, \pi(t)\rangle, \quad t \in [0,T]. \end{cases} \tag{1.7}$$

Thus, $(Y(\cdot), \pi(\cdot))$ is an adapted solution of BSDE (1.2). Consequently, under (1.3), if $r(\cdot)$ and $\theta(\cdot)$ are bounded, by [3,13], we know that for any $\xi \in L^p_{\mathcal{F}_T}(\Omega;\mathbb{R})$, the set of all $\mathcal{F}_T$-measurable and L^p-integrable random variables (with $p > 1$), BSDE (1.5) admits a unique adapted solution $(Y(\cdot), Z(\cdot))$ (in certain spaces). Thus, by (1.6)–(1.7), BSDE (1.2) admits an adapted solution $(Y(\cdot), \pi(\cdot))$, and the market is complete (in a suitable sense).

Note that (1.3) is equivalent to

$$\mathcal{N}\big(\sigma(t)\big) = \mathcal{R}\big(\sigma(t)^T\big)^{\perp} = \{0\}, \quad t \in [0,T], \text{ a.s.}, \tag{1.8}$$

where $\mathcal{N}(A)$ and $\mathcal{R}(A)$ stand for the kernel and the range of the matrix A. Unless $n = d$, condition (1.8) is irrelevant to the following *range condition*:

$$b(t) - r(t)\mathbf{1} \in \mathcal{R}\big(\sigma(t)\big), \quad t \in [0,T], \text{ a.s.}, \tag{1.9}$$

which implies the existence of a process $\bar\theta(\cdot)$, called risk premium, such that

$$b(t) - r(t)\mathbf{1} = \sigma(t)\bar\theta(t), \quad t \in [0,T], \text{ a.s.}$$

It is known that when $\bar\theta(\cdot)$ exists and has some integrability, the market is arbitrage-free [6]. Now, if both (1.3) (or (1.8)) and (1.9) hold, it is necessarily that (note (1.4))

$$\bar\theta(t) = \big[\sigma(t)^T\sigma(t)\big]^{-1}\sigma(t)^T\big[b(t) - r(t)\mathbf{1}\big] \equiv \theta(t), \quad t \in [0,T], \text{ a.s.}$$

We point out here that, one may have (1.3) without the existence of a risk premium.

From the above analysis, we see that under (1.3), in order to study the completeness of the market, it suffices to look at the solvability of BSDE (1.5). Note that in general, $\theta(\cdot)$ defined by (1.4) is not necessarily bounded. On the other hand, the interest rate $r(\cdot)$ might be unbounded (if it follows some short rate term structure stochastic differential equation, such as Vasicek's model, CIR model, or Hull–White model, etc. [12]). Hence, (1.5) could be a linear BSDE with unbounded coefficients. It is interesting to know under what conditions and in what sense, such kind of linear BSDEs are solvable. Correspondingly, in what sense, the market is complete.

Let us briefly explain our basic idea for solving linear BSDE (1.5) (with possibly unbounded coefficients). Suppose $(Y(\cdot), Z(\cdot))$ is an adapted solution of (1.5). Let $M(\cdot)$ be the solution of the following SDE:

$$\begin{cases} dM(t) = -M(t)r(t)\,dt - M(t)\langle\theta(t), dW(t)\rangle, \quad t \geqslant 0, \\ M(0) = 1, \end{cases} \tag{1.10}$$

which is given by

$$M(t) = e^{-\int_0^t [r(s) + \frac{1}{2}|\theta(s)|^2]\,ds - \int_0^t \langle\theta(s), dW(s)\rangle}, \quad t \in [0,T]. \tag{1.11}$$

We will call $M(\cdot)$ the exponential process corresponding to $(r(\cdot),\theta(\cdot))$, which is a generalization of the so-called exponential super-martingale [5,7]. Applying Itô's formula to $M(\cdot)Y(\cdot)$, we obtain

$$d\big[M(t)Y(t)\big]=M(t)\big\langle Z(t)-Y(t)\theta(t),dW(t)\big\rangle,\quad t\in[0,T].$$

Thus, if we denote

$$\begin{cases}\widetilde{Y}(t)=M(t)Y(t),\\ \widetilde{Z}(t)=M(t)[Z(t)-Y(t)\theta(t)],\end{cases}\quad t\in[0,T],\tag{1.12}$$

then

$$\begin{cases}d\widetilde{Y}(t)=\langle\widetilde{Z}(t),dW(t)\rangle,\quad t\in[0,T],\\ \widetilde{Y}(T)=M(T)\xi\equiv\tilde{\xi}.\end{cases}\tag{1.13}$$

Clearly, (1.13) is one of the simplest BSDEs. If $\tilde{\xi}$ has some integrability, we should have a unique adapted solution $(\widetilde{Y}(\cdot),\widetilde{Z}(\cdot))$ (in a suitable space). Then we could define

$$\begin{cases}Y(t)=M(t)^{-1}\widetilde{Y}(t),\\ Z(t)=M(t)^{-1}[\widetilde{Z}(t)+\widetilde{Y}(t)\theta(t)],\end{cases}\quad t\in[0,T].\tag{1.14}$$

One expects that $(Y(\cdot),Z(\cdot))$ defined by (1.14) should give the adapted solution to BSDE (1.5). Now the questions left are the following: Under what conditions on $r(\cdot)$, $\theta(\cdot)$, and ξ, processes $Y(\cdot)$ and $Z(\cdot)$ can be well-defined by (1.14)? What are the spaces that the adapted solution $(Y(\cdot),Z(\cdot))$ belong to? Once these questions are answered, we will obtain the completeness of the market in some proper sense. It is clear that the key to answer the above questions is to estimate $M(T)$ and $M(\cdot)^{-1}$, which is the main part of the current paper.

The rest of the paper is organized as follows. In Section 2, we introduce some basic spaces which will be used in later sections. Section 3 is devoted to various estimates of $M(\cdot)$ which we call it an exponential process. In Section 4, we discuss the solvability of BSDE (1.5), which leads to the completeness of the markets for the case that (1.3) holds. Finally, two illustrative examples are presented in Section 5.

2. Some spaces

Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geqslant 0},\mathbf{P})$ be a complete filtered probability space satisfying the usual condition [5,14], on which a d-dimensional standard Brownian motion $W(\cdot)$ is defined with $\{\mathcal{F}_t\}_{t\geqslant 0}$ being the natural filtration of $W(\cdot)$ augmented by all the $\mathbf{P}$-null sets in $\mathcal{F}$. Let H be any finite dimensional Euclidean space (such as $\mathbb{R}^n$, $\mathbb{R}^{m\times n}$, and so on) whose norm is denoted by $|\cdot|$. Let $L^0_{\mathcal{F}_T}(\Omega;H)$ be the set of all $\mathcal{F}_T$-measurable H-valued random variables (no integrability is assumed). Introduce the following spaces:

$$L^p_{\mathcal{F}_T}(\Omega;H)\triangleq\big\{\xi:\Omega\to H\ \big|\ \xi\in L^0_{\mathcal{F}_T}(\Omega;H),\ E|\xi|^p<\infty\big\},\quad p\in(0,\infty).$$

The space $L^\infty_{\mathcal{F}_T}(\Omega;H)$ can be defined similarly. For $p\in[1,\infty]$, $L^p_{\mathcal{F}_T}(\Omega;H)$ is a Banach space, and for $p\in(0,1)$, it is a complete metric space.

Next, let $L^0_{\mathcal{F}}(0,T;H)$ be the set of all $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted processes $\varphi:[0,T]\to H$ (no any integrability is assumed here), and let

$$\begin{cases} L^p_{\mathcal{F}}(\Omega;L^q(0,T;H)) \\ \quad \triangleq \big\{\varphi:[0,T]\times\Omega\to H \mid \varphi(\cdot)\in L^0_{\mathcal{F}}(0,T;H),\ E\big[\int_0^T|\varphi(t)|^q\,dt\big]^{p/q}<\infty\big\}, \\ \quad 0<p,q<\infty, \\ L^p_{\mathcal{F}}(0,T;H)\triangleq L^p_{\mathcal{F}}(\Omega;L^p(0,T;H)), \quad 0<p<\infty. \end{cases} \tag{2.1}$$

Spaces $L^\infty_{\mathcal{F}}(\Omega;L^p(0,T;H))$, $L^p_{\mathcal{F}}(\Omega;L^\infty(0,T;H))$ $(0<p<\infty)$, and $L^\infty_{\mathcal{F}}(0,T;H)\equiv L^\infty_{\mathcal{F}}(\Omega;L^\infty(0,T;H))$ can be defined in an obvious way. It is clear that for any $1\leqslant p,q\leqslant\infty$, $L^p_{\mathcal{F}}(\Omega;L^q(0,T;H))$ is a Banach space, and for either $p\in(0,1)$ or $q\in(0,1)$, $L^p_{\mathcal{F}}(\Omega;L^q(0,T;H))$ is a complete metric space.

Further, we define

$$\begin{cases} L^0_{\mathcal{F}}(\Omega;C([0,T];H))\triangleq\{\varphi:[0,T]\times\Omega\to H \mid \varphi(\cdot)\in L^0_{\mathcal{F}}(0,T;H), \\ \qquad\qquad\qquad\qquad\qquad \text{almost all paths of } \varphi(\cdot) \text{ are continuous}\}, \\ L^p_{\mathcal{F}}(\Omega;C([0,T];H))\triangleq\{\varphi:[0,T]\times\Omega\to H \mid \varphi(\cdot)\in L^0_{\mathcal{F}}(\Omega;C([0,T];H)), \\ \qquad\qquad\qquad\qquad\qquad E[\sup_{t\in[0,T]}|\varphi(t)|^p]<\infty\}, \quad p>0. \end{cases}$$

The space $L^\infty_{\mathcal{F}}(\Omega;C([0,T];H))$ can be defined similarly. For $p\in[1,\infty]$, $L^p_{\mathcal{F}}(\Omega;C([0,T];H))$ is a Banach space and for $p\in(0,1)$, it is a complete linear metric space.

Likewise, we define (compare with (2.1))

$$L^q_{\mathcal{F}}\big(0,T;L^p(\Omega;H)\big)\triangleq\left\{\varphi:[0,T]\times\Omega\to H \;\Big|\; \varphi(\cdot)\in L^0_{\mathcal{F}}(0,T;H),\ \int_0^T\big[E\big|\varphi(t)\big|^p\big]^{q/p}\,dt<\infty\right\}, \quad 0<p,q<\infty. \tag{2.2}$$

Again, spaces $L^\infty_{\mathcal{F}}(0,T;L^p(\Omega;H))$ and $L^p_{\mathcal{F}}(0,T;L^\infty(\Omega;H))$ (with $0<p<\infty$) can be defined similarly. It is also clear that $L^q_{\mathcal{F}}(0,T;L^p(\Omega;H))$ is a Banach space, for any $1\leqslant p,q\leqslant\infty$, and if $p\in(0,1)$ or $q\in(0,1)$, $L^q_{\mathcal{F}}(\Omega;L^p(0,T;H))$ is a complete metric space.

By definition (see (2.1) and (2.2)), we see that

$$L^p_{\mathcal{F}}\big(0,T;L^p(\Omega;H)\big)=L^p_{\mathcal{F}}\big(\Omega;L^p(0,T;H)\big)\triangleq L^p_{\mathcal{F}}(0,T;H), \quad 0<p\leqslant\infty.$$

Using Hölder's inequality, one has that

$$\begin{cases} L^\infty_{\mathcal{F}}(\Omega;C([0,T];H))\subseteq L^\infty_{\mathcal{F}}(0,T;H)\subseteq L^p_{\mathcal{F}}(\Omega;L^q(0,T;H)), \\ L^\infty_{\mathcal{F}}(\Omega;C([0,T]);H))\subseteq L^p_{\mathcal{F}}(\Omega;C([0,T];H))\subseteq L^p_{\mathcal{F}}(\Omega;L^q(0,T;H)), \\ \quad 0<p,q\leqslant\infty. \end{cases}$$

Further, it follows from a similar proof of Young's inequality that

$$\begin{cases} L^p_{\mathcal{F}}(\Omega;L^q(0,T;H))\subseteq L^q_{\mathcal{F}}(0,T;L^p(\Omega;H)), \quad 0<p\leqslant q\leqslant\infty, \\ L^q_{\mathcal{F}}(0,T;L^p(\Omega;H))\subseteq L^p_{\mathcal{F}}(\Omega;L^q(0,T;H)), \quad 0<q\leqslant p\leqslant\infty. \end{cases}$$

In what follows, we denote

$$\begin{aligned} L^{q+}_{\mathcal{F}_T}(\Omega;H) &= \bigcup_{p\in(q,\infty]} L^p_{\mathcal{F}_T}(\Omega;H), \\ L^{q-}_{\mathcal{F}_T}(\Omega;H) &= \bigcap_{p\in(0,q)} L^p_{\mathcal{F}_T}(\Omega;H), \quad \forall q>0. \end{aligned} \tag{2.3}$$

In a same fashion, we can define $L^{p\pm}_{\mathcal{F}}(\Omega;L^{q\pm}(0,T;H))$, $L^{p\pm}_{\mathcal{F}}(\Omega;C([0,T];H))$, and $C_{\mathcal{F}}([0,T];L^{p\pm}(\Omega;H))$, and so on.

Finally, for given adapted processes $b(\cdot)$, $r(\cdot)$ and $\sigma(\cdot)$, we define

$$\begin{aligned} \Pi^p[0,T] \triangleq \big\{\pi(\cdot)\in L^0_{\mathcal{F}}\big(0,T;\mathbb{R}^n\big) \,\big|\, \big\langle b(\cdot)-r(\cdot)\mathbf{1}\big\rangle \in L^p_{\mathcal{F}}\big(\Omega;L^1(0,T;\mathbb{R})\big), \\ \sigma(\cdot)^T\pi(\cdot)\in L^p_{\mathcal{F}}\big(\Omega;L^2\big(0,T;\mathbb{R}^d\big)\big)\big\}, \quad p>0. \end{aligned}$$

Note that for any $\pi(\cdot)\in\Pi^p[0,T]$, the right-hand side of the SDE in (1.2) makes sense. Similar to (2.3), we may define $\Pi^{p\pm}[0,T]$.

3. Exponential processes

Let $r(\cdot)\in L^1_{\mathcal{F}}(0,T;\mathbb{R})$ and $\theta(\cdot)\in L^1_{\mathcal{F}}(\Omega;L^2(0,T;\mathbb{R}^d))$ be given. We define the following process, call it the *exponential process* corresponding to $r(\cdot)$ and $\theta(\cdot)$ (see (1.11)):

$$M\big(t;r(\cdot),\theta(\cdot)\big) = e^{-\int_0^t [r(s)+\frac{1}{2}|\theta(s)|^2]\,ds - \int_0^t \langle\theta(s),dW(s)\rangle}, \quad t\in[0,T]. \tag{3.1}$$

The above is a well-defined $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted process with continuous paths. Therefore, $M(\cdot\,;r(\cdot),\theta(\cdot))\in L^0_{\mathcal{F}}(\Omega;C([0,T];\mathbb{R}))$. Moreover, by Itô's formula, it satisfies SDE (1.10). From Section 1, we see that $M(\cdot\,;r(\cdot),\theta(\cdot))$ plays a crucial role in solving linear BSDE (1.5) (with possibly unbounded coefficients). The purpose of this section is to provide various estimates for this process, under various integrability conditions on $r(\cdot)$ and $\theta(\cdot)$. The results obtained in this section will lead to the solvability of linear BSDEs with possibly unbounded coefficients, in proper spaces, which will lead to the completeness of the corresponding markets.

First of all, by a standard argument, due to the integrability of $r(\cdot)$ and $\theta(\cdot)$, we can find a sequence of stopping times τ_k such that $\tau_k\to T$ as $k\to\infty$ and for any $p\in[1,\infty)$ and $k\geqslant 1$,

$$E\Big[\sup_{t\in[0,\tau_k]} M\big(t;r(\cdot),\theta(\cdot)\big)^p\Big] \leqslant C_{k,p}, \tag{3.2}$$

for some constant $C_{k,p}>0$ depending on k and p. However, in general, we are not able to claim $M(\cdot\,;r(\cdot),\theta(\cdot))\in L^p_{\mathcal{F}}(0,T;\mathbb{R})$, for any $p>0$. Here is a simple example.

Example 3.1. Let $W(\cdot)$ be a one-dimensional standard Brownian motion. By Itô's formula, we know that

$$W(t)^4 = 4\int_0^t W(s)^3\,dW(s) + 6\int_0^t W(s)^2\,ds, \quad t\in[0,T].$$

Thus, by taking

$$\theta(t) = -4W(t)^3, \quad r(t) = -8\big[W(t)^6 + 6W(t)^2\big], \quad t \in [0, T],$$

we have $r(\cdot), \theta(\cdot) \in L^{\infty-}_{\mathcal{F}}(0, T; \mathbb{R}) \setminus L^{\infty}_{\mathcal{F}}(0, T; \mathbb{R})$, and

$$M\big(t; r(\cdot), \theta(\cdot)\big) = e^{W(t)^4}, \quad t \in [0, T].$$

The following computation is straightforward: For any $t > 0$, and any $p > 0$,

$$E\big[M\big(t; r(\cdot), \theta(\cdot)\big)^p\big] = E\big[e^{pW(t)^4}\big] = \sum_{k=0}^{\infty} \frac{E\{[pW(t)]^{4k}\}}{k!} = \sum_{k=0}^{\infty} \frac{(4k)!(pt^2)^k}{2^k(2k)!k!} = \infty.$$

In the above, we have used the fact that

$$E\big[W(t)^{2k}\big] = \frac{(2k)!t^k}{2^k k!}, \quad t \geqslant 0,\ k \geqslant 1,$$

which can be proved by induction. Hence, in particular, for any $p > 0$, $M(T; r(\cdot), \theta(\cdot)) \notin L^p_{\mathcal{F}_T}(\Omega; \mathbb{R})$ (for the above case).

We now look at the integrability of $M(\cdot\,; r(\cdot), \theta(\cdot))$ and $M(T; r(\cdot), \theta(\cdot))$, under certain integrable conditions on $r(\cdot)$ and $\theta(\cdot)$. Some relevant study can be found in [17,19].

Note that $M(\cdot\,; 0, \theta(\cdot))$ is the so-called exponential super-martingale (see [5,7]), provided $\theta(\cdot) \in L^1_{\mathcal{F}}(\Omega; L^2(0, T; \mathbb{R}^d))$. In this case, the following holds:

$$\sup_{t\in[0,T]} E\big[M\big(t; 0, \theta(\cdot)\big)\big] \leqslant 1. \tag{3.3}$$

This fact will be used several times below. We now state and prove our first result concerning the estimates of exponential process $M(\cdot\,; r(\cdot), \theta(\cdot))$.

Theorem 3.2. *Let $r(\cdot) \in L^1_{\mathcal{F}}(0, T; \mathbb{R})$ and $\theta(\cdot) \in L^1_{\mathcal{F}}(\Omega; L^2(0, T; \mathbb{R}^d))$.*

(i) *Suppose the following holds*:

$$\sup_{t\in[0,T]} E\big[e^{-\alpha\int_0^t r(s)\,ds}\big] < \infty, \tag{3.4}$$

for some $\alpha > 0$. Then

$$\sup_{t\in[0,T]} E\big[M\big(t; r(\cdot), \theta(\cdot)\big)^{\frac{\alpha}{\alpha+1}}\big] \leqslant \Big\{\sup_{t\in[0,T]} E\big[e^{-\alpha\int_0^t r(s)\,ds}\big]\Big\}^{\frac{1}{\alpha}}. \tag{3.5}$$

(ii) *Suppose* (3.4) *holds for some $\alpha > 0$ and*

$$E\big[e^{\frac{\beta}{2}\int_0^T |\theta(s)|^2\,ds}\big] < \infty, \tag{3.6}$$

for some $\beta > 1$. Then

$$\begin{aligned} &\sup_{t\in[0,T]} E\big[M\big(t; r(\cdot), \theta(\cdot)\big)^{\frac{\alpha\beta}{\beta+\alpha(2\sqrt{\beta}-1)}}\big] \\ &\quad\leqslant \Big\{E\big[e^{\frac{\beta}{2}\int_0^T |\theta(s)|^2\,ds}\big]\Big\}^{\frac{\alpha(\sqrt{\beta}-1)}{\beta+\alpha(2\sqrt{\beta}-1)}} \Big\{\sup_{t\in[0,T]} E\big[e^{-\alpha\int_0^t r(s)\,ds}\big]\Big\}^{\frac{\beta}{\beta+\alpha(2\sqrt{\beta}-1)}}. \end{aligned} \tag{3.7}$$

(iii) *Let*

$$\inf_{t\in[0,T]}\int_0^t r(s)\,ds \geqslant -\bar{a}, \quad a.s., \tag{3.8}$$

for some $\bar{a}\in\mathbb{R}$, *then*

$$\sup_{t\in[0,T]} E\big[M\big(t;r(\cdot),\theta(\cdot)\big)^p\big] \leqslant e^{\bar{a}p}, \quad \forall p\in(0,1). \tag{3.9}$$

If (3.6) *holds for some* $\beta>1$, *then*

$$\sup_{t\in[0,T]} E\big[M\big(t;r(\cdot),\theta(\cdot)\big)^{\frac{\beta}{2\sqrt{\beta}-1}}\big] \leqslant e^{\frac{\bar{a}\beta}{2\sqrt{\beta}-1}}\big\{E\big[e^{\frac{\beta}{2}\int_0^T|\theta(s)|^2\,ds}\big]\big\}^{\frac{\sqrt{\beta}-1}{2\sqrt{\beta}-1}}, \tag{3.10}$$

and if (3.6) *holds only for* $\beta=1$, *then* (3.9) *holds for* $p=1$.

Let us make some comments on the above results. For convenience, suppose $r(\cdot)$ is bounded from below, then (3.8) (and of course (3.4)) holds. For such a case, regardless of conditions on $\theta(\cdot)$ (as long as it is in $L^1_{\mathcal{F}}(\Omega;L^2(0,T;\mathbb{R}^d))$), we have (3.5) which means

$$M\big(\cdot\,;r(\cdot),\theta(\cdot)\big)\in L^\infty_{\mathcal{F}}\big(0,T;L^{\frac{\alpha}{\alpha+1}}(\Omega;\mathbb{R})\big). \tag{3.11}$$

Since $\frac{\alpha}{\alpha+1}<1$, estimate (3.11) is not very strong. Next, if we have some additional conditions on $\theta(\cdot)$, namely, (3.6) holds for some $\beta>1$, then due to

$$\frac{\alpha\beta}{\beta+\alpha(2\sqrt{\beta}-1)}>\frac{\alpha}{\alpha+1},$$

the conclusion

$$M\big(\cdot\,;r(\cdot),\theta(\cdot)\big)\in L^\infty_{\mathcal{F}}\big(0,T;L^{\frac{\alpha\beta}{\beta+\alpha(2\sqrt{\beta}-1)}}(\Omega;\mathbb{R})\big)$$

drawn from (3.7) is an improvement of (3.11). We know that for any $\beta>1$, the map $\alpha\mapsto\frac{\alpha\beta}{\beta+\alpha(2\sqrt{\beta}-1)}$ is strictly increasing and when $\alpha>\frac{\beta}{(\sqrt{\beta}-1)^2}$,

$$\frac{\alpha\beta}{\beta+\alpha(2\sqrt{\beta}-1)}>1.$$

Further,

$$\lim_{\alpha\to\infty}\frac{\alpha\beta}{\beta+\alpha(2\sqrt{\beta}-1)}=\frac{\beta}{2\sqrt{\beta}-1}.$$

Thus, (3.10) means that when $\alpha=\infty$, one has

$$M\big(\cdot\,;r(\cdot),\theta(\cdot)\big)\in L^\infty_{\mathcal{F}}\big(0,T;L^{\frac{\beta}{2\sqrt{\beta}-1}}(\Omega;\mathbb{R})\big).$$

Proof. (i) Let $p>0$, $\gamma>1$. We observe (note (3.3)):

$$\begin{aligned}&E\big[M\big(t;r(\cdot),\theta(\cdot)\big)^p\big]\\&\quad=E\big[e^{-p\int_0^t[r(s)+\frac{1}{2}|\theta(s)|^2]\,ds-p\int_0^t\langle\theta(s),dW(s)\rangle}\big]\end{aligned}$$

$$\leqslant \big\{E\big[M\big(t;0,p\gamma\theta(\cdot)\big)\big]\big\}^{\frac{1}{\gamma}}\big\{E\big[e^{\frac{\gamma}{\gamma-1}\int_0^t[\frac{p^2\gamma-p}{2}|\theta(s)|^2-pr(s)]\,ds}\big]\big\}^{\frac{\gamma-1}{\gamma}}$$
$$\leqslant \big\{E\big[e^{\frac{p^2\gamma^2-p\gamma}{2(\gamma-1)}\int_0^t|\theta(s)|^2\,ds-\frac{p\gamma}{\gamma-1}\int_0^t r(s)\,ds}\big]\big\}^{\frac{\gamma-1}{\gamma}}. \tag{3.12}$$

Thus, for $p=\frac{\alpha}{\alpha+1}<1$, we take $\gamma=\frac{\alpha+1}{\alpha}>1$. Then $p\gamma=1$, $\frac{p\gamma}{\gamma-1}=\alpha$, and the above gives (3.5).

(ii) Next, we assume $\beta>1$. For any $q>1$, by (3.12), we have

$$E\big[M\big(t;r(\cdot),\theta(\cdot)\big)^p\big]$$
$$\leqslant \big\{E\big[e^{\frac{p^2\gamma^2-p\gamma}{2(\gamma-1)}\int_0^t|\theta(s)|^2\,ds-\frac{p\gamma}{\gamma-1}\int_0^t r(s)\,ds}\big]\big\}^{\frac{\gamma-1}{\gamma}}$$
$$\leqslant \big\{E\big[e^{\frac{q(p^2\gamma^2-p\gamma)}{2(\gamma-1)}\int_0^t|\theta(s)|^2\,ds}\big]\big\}^{\frac{\gamma-1}{q\gamma}}\big\{E\big[e^{-\frac{pq\gamma}{(q-1)(\gamma-1)}\int_0^t r(s)\,ds}\big]\big\}^{\frac{(q-1)(\gamma-1)}{q\gamma}}. \tag{3.13}$$

Now, we take

$$\begin{cases} p=\frac{\alpha\beta}{\beta+\alpha(2\sqrt{\beta}-1)}>\frac{\alpha}{\alpha+1},\\ \gamma=\frac{\beta+\alpha(2\sqrt{\beta}-1)}{\alpha\sqrt{\beta}}\equiv\frac{\sqrt{\beta}}{p}>\frac{1}{p},\\ q=\frac{\beta+\alpha(\sqrt{\beta}-1)}{\alpha(\sqrt{\beta}-1)}>1. \end{cases}$$

Then

$$\begin{cases} \frac{q(p^2\gamma^2-p\gamma)}{\gamma-1}=\beta, & \frac{pq\gamma}{(q-1)(\gamma-1)}=\alpha,\\ \frac{\gamma-1}{q\gamma}=\frac{\alpha(\sqrt{\beta}-1)}{\beta+\alpha(2\sqrt{\beta}-1)}, & \frac{(q-1)(\gamma-1)}{q\gamma}=\frac{\beta}{\beta+\alpha(2\sqrt{\beta}-1)}. \end{cases}$$

With these choices, (3.7) follows from (3.13).

(iii) Let (3.8) holds. For any $p\in(0,1)$, take $\gamma=\frac{1}{p}>1$. By (3.12), we obtain (3.9). Next, we take

$$p=\frac{\beta}{2\sqrt{\beta}-1}>1,\qquad \gamma=\frac{2\sqrt{\beta}-1}{\sqrt{\beta}}\equiv\frac{\sqrt{\beta}}{p}>\frac{1}{p}.$$

Then

$$\frac{p^2\gamma^2-p\gamma}{\gamma-1}=\beta,\qquad \frac{p\gamma}{\gamma-1}=\frac{\beta}{\sqrt{\beta}-1},\qquad \frac{\gamma-1}{\gamma}=\frac{\sqrt{\beta}-1}{2\sqrt{\beta}-1},$$

and (3.10) follows from (3.12). Finally, for $\beta=1$, by taking $p=1$ in (3.12), using (3.3) and (3.8), we obtain (3.9) for $p=1$. □

Note that when $r(\cdot)=0$, we can take $\bar{a}=0$. Then (3.10) reads

$$\sup_{t\in[0,T]}E\big[M\big(t;0,\theta(\cdot)\big)^{\frac{\beta}{2\sqrt{\beta}-1}}\big]\leqslant\big\{E\big[e^{\frac{\beta}{2}\int_0^T|\theta(s)|^2\,ds}\big]\big\}^{\frac{\sqrt{\beta}-1}{2\sqrt{\beta}-1}}. \tag{3.14}$$

This result is equivalent to that in [15].

Example 3.1 shows that for some $r(\cdot),\theta(\cdot)\in L_{\mathcal{F}}^{\infty-}(0,T;\mathbb{R})$, with $r(\cdot)$ suitably chosen according to $\theta(\cdot)$, the exponential process $M(\cdot\,;r(\cdot),\theta(\cdot))\notin L_{\mathcal{F}}^p(0,T;\mathbb{R})$, for all $p>0$. The following result shows that even for $r(\cdot)=0$ (i.e., without any "help" from $r(\cdot)$),

$M(\cdot\,;0,\theta(\cdot))$ could also be not in $L^p_{\mathcal{F}}(0,T;\mathbb{R})$ for some $p>1$ (although (3.3) still holds). In some sense, this shows the necessity of condition (3.6) for (3.7).

Proposition 3.3. *Let $W(\cdot)$ be a one-dimensional standard Brownian motion. Then for any $T>0$,*

$$\sup_{t\in[0,T]} E\big[M\big(t;0,-\gamma W(\cdot)\big)^4\big]=\infty,\quad \forall\gamma\geqslant\frac{1}{T}. \tag{3.15}$$

Proof. For any $q>p>0$, consider the following:

$$\begin{aligned}E\big[e^{p\gamma\int_0^T W(t)\,dW(t)}\big]&=E\big[M\big(T;0,-\gamma W(\cdot)\big)^p e^{\frac{p\gamma^2}{2}\int_0^T W(t)^2\,dt}\big]\\&\leqslant\big\{E\big[M\big(T;0,-\gamma W(\cdot)\big)^q\big]\big\}^{\frac{p}{q}}\big\{E\big[e^{\frac{pq\gamma^2}{2(q-p)}\int_0^T W(t)^2\,dt}\big]\big\}^{\frac{q-p}{q}}.\end{aligned}$$

Note that

$$\begin{aligned}&E\big[e^{p\gamma\int_0^T\langle W(t),dW(t)\rangle}\big]\\&\quad=e^{-\frac{p\gamma T}{2}}\sum_{k=0}^{\infty}\frac{(p\gamma)^k}{2^k k!}E\big[W(T)^{2k}\big]=e^{-\frac{p\gamma T}{2}}\sum_{k=0}^{\infty}\frac{(p\gamma)^k}{2^k k!}\frac{(2k)!}{2^k k!}T^k\\&\quad\sim e^{-\frac{p\gamma T}{2}}\sum_{k=0}^{\infty}\frac{(p\gamma T)^k}{4^k}\frac{(2k)^{2k}\sqrt{4k\pi}}{e^{2k}}\frac{e^{2k}}{k^{2k}2k\pi}=e^{-\frac{p\gamma T}{2}}\sum_{k=0}^{\infty}\frac{(p\gamma T)^k}{\sqrt{k\pi}}.\end{aligned}\tag{3.16}$$

Thus, the left-hand side of (3.16) is finite if and only if $p\gamma T<1$. On the other hand, by Stirling formula,

$$\begin{aligned}&E\big[e^{\frac{pq\gamma^2}{2(q-p)}\int_0^T W(t)^2\,dt}\big]\\&\quad=\sum_{k=0}^{\infty}\bigg(\frac{pq\gamma^2}{2(q-p)}\bigg)^k\frac{E[\int_0^T W(t)^2\,dt]^k}{k!}\\&\quad\leqslant 1+\sum_{k=1}^{\infty}\bigg(\frac{pq\gamma^2T^2}{q-p}\bigg)^k\frac{(2k)!}{(k+1)4^k(k!)^2}\sim 1+\sum_{k=1}^{\infty}\bigg(\frac{pq\gamma^2T^2}{q-p}\bigg)^k\frac{1}{(k+1)\sqrt{k\pi}}.\end{aligned}\tag{3.17}$$

Hence, the left-hand side of (3.17) is finite if and only if $\frac{pq\gamma^2T^2}{q-p}\leqslant 1$. Now, we let

$$0<\gamma T<1,\qquad p=\frac{1}{\gamma T},\qquad q=\frac{1}{\gamma T(1-\gamma T)}.$$

Then

$$p\gamma T=1,\qquad \frac{pq\gamma^2T^2}{q-p}=1.$$

Hence by the above analysis, we have

$$E\big[M\big(T;0,-\gamma W(\cdot)\big)^{\frac{1}{\gamma T(1-\gamma T)}}\big]=\infty,\quad \forall 0<\gamma T<1. \tag{3.18}$$

Since $\lambda \mapsto \frac{1}{\lambda(1-\lambda)}$ attains its minimum over $(0,1)$ at $\lambda = \frac{1}{2}$, we see that

$$E\left[M\left(\frac{1}{\gamma}; 0, -\gamma W(\cdot)\right)^4\right] = \infty, \quad \forall \gamma > 0.$$

Thus, (3.15) follows. $\square$

Note that for $\theta(\cdot) = -\gamma W(\cdot)$ on $[0,T]$ (with $0 < \gamma T < 1$), similar to (3.17), we have

$$E\left[e^{\frac{\beta}{2}\int_0^T |\theta(s)|^2\,ds}\right] < \infty \quad \Longleftrightarrow \quad \beta \leqslant \frac{1}{\gamma^2 T^2}.$$

Let us take the best possibility: $\beta = \frac{1}{\gamma^2 T^2} > 1$. Then

$$\frac{\beta}{2\sqrt{\beta} - 1} = \frac{1}{\gamma T(2 - \gamma T)}.$$

Hence, (3.14) implies

$$E\left[M\big(T; 0, -\gamma W(\cdot)\big)^{\frac{1}{\gamma T(2-\gamma T)}}\right] < \infty, \quad \forall 0 < \gamma T < 1.$$

Comparing the above with (3.18), we see that it is not clear whether $E[M(T; 0, -\gamma W(\cdot))^p]$ is finite for $\frac{1}{\gamma T(2-\gamma T)} < p < \frac{1}{\gamma T(1-\gamma T)}$.

More generally, it will be interesting to find

$$p_0\big(T, r(\cdot), \theta(\cdot)\big) \triangleq \inf\Big\{p > 0 \;\Big|\; \sup_{t\in[0,T]} E\big[M\big(t; r(\cdot), \theta(\cdot)\big)^p\big] = \infty\Big\}.$$

The answer to this problem is unknown to us at this moment.

We note that Theorem 3.2 says that under proper conditions, the exponential process $M(\cdot\,; r(\cdot), \theta(\cdot)) \in L^\infty_{\mathcal{F}}(0, T; L^p(\Omega; \mathbb{R}))$ for some $p > 0$. The following result says that if we assume stronger conditions on $r(\cdot)$ and $\theta(\cdot)$, then $M(\cdot\,; r(\cdot), \theta(\cdot)) \in L^p_{\mathcal{F}}(\Omega; C([0,T]; \mathbb{R}))$ for some $p > 0$, which is a subspace of $L^\infty_{\mathcal{F}}(0, T; L^p(\Omega; \mathbb{R}))$ (see relevant results in Section 2). Hence, the following theorem is a refinement of Theorem 3.2.

Theorem 3.4.

(i) *Suppose* (3.6) *holds for some* $\beta > 1$, *and*

$$E\Big[\sup_{t\in[0,T]} e^{-\alpha\int_0^t r(s)\,ds}\Big] < \infty, \tag{3.19}$$

for some $\alpha > 0$. *Then* $M(\cdot\,; r(\cdot), \theta(\cdot)) \in L_{\mathcal{F}}^{\frac{\alpha\beta}{\beta+\alpha(2\sqrt{\beta}-1)}}(\Omega; C([0,T]; \mathbb{R}))$ *and*

$$\begin{aligned} &E\Big[\sup_{t\in[0,T]} M\big(t; r(\cdot), \theta(\cdot)\big)^{\frac{\alpha\beta}{\beta+\alpha(2\sqrt{\beta}-1)}}\Big] \\ &\quad \leqslant C\Big\{E\big[e^{\frac{\beta}{2}\int_0^T |\theta(s)|^2\,ds}\big]\Big\}^{\frac{\alpha(\sqrt{\beta}-1)}{\beta+\alpha(2\sqrt{\beta}-1)}} \Big\{E\Big[\sup_{t\in[0,T]} e^{-\alpha\int_0^t r(s)\,ds}\Big]\Big\}^{\frac{\beta}{\beta+\alpha(2\sqrt{\beta}-1)}}, \end{aligned} \tag{3.20}$$

with $C = \big(\frac{\beta}{(\sqrt{\beta}-1)^2}\big)^{\frac{\alpha\beta}{\beta+\alpha(2\sqrt{\beta}-1)}}$.

(ii) *Suppose* (3.6) *holds for some* $\beta \in (0,1]$, *and* (3.19) *holds for some* $\alpha > 0$. *Then* $M(\cdot\,; r(\cdot), \theta(\cdot)) \in L_{\mathcal{F}}^{\frac{\alpha\sqrt{\beta}}{\alpha+\sqrt{\beta}}-}(\Omega; C([0,T];\mathbb{R}))$, *and for any* $p \in \big(\frac{\alpha\sqrt{\beta}}{2\alpha+\sqrt{\beta}}, \frac{\alpha\sqrt{\beta}}{\alpha+\sqrt{\beta}}\big)$,

$$E\Big[\sup_{t\in[0,T]} M\big(t; r(\cdot), \theta(\cdot)\big)^p\Big] \leqslant C\big\{E\big[e^{\frac{\beta}{2}\int_0^T |\theta(s)|^2\,ds}\big]\big\}^{1-\frac{\alpha p}{(\alpha-p)\sqrt{\beta}}} \Big\{E\Big[\sup_{t\in[0,T]} e^{-\alpha\int_0^t r(s)\,ds}\Big]\Big\}^{\frac{p}{\alpha}}, \tag{3.21}$$

with $C = \big(\frac{\alpha^2 p^2}{[\alpha\sqrt{\beta}-p(\alpha+\sqrt{\beta})]^2}\big)^{\frac{\alpha p^2}{\sqrt{\beta}[(2\alpha+\sqrt{\beta})p-\alpha\sqrt{\beta}]}}$.

(iii) *Let* (3.8) *hold for some* $\bar{a} \in \mathbb{R}$. *Then, in the case that* (3.6) *holds with* $\beta > 1$, $M(\cdot\,; r(\cdot), \theta(\cdot)) \in L_{\mathcal{F}}^{\frac{\beta}{2\sqrt{\beta}-1}}(\Omega; C([0,T];\mathbb{R}))$ *and* (3.20) *becomes*

$$E\Big[\sup_{t\in[0,T]} M\big(t; r(\cdot), \theta(\cdot)\big)^{\frac{\beta}{2\sqrt{\beta}-1}}\Big] \leqslant C\big\{E\big[e^{\frac{\beta}{2}\int_0^T |\theta(s)|^2\,ds}\big]\big\}^{\frac{\sqrt{\beta}-1}{2\sqrt{\beta}-1}}, \tag{3.22}$$

with $C = e^{\bar{a}p}\big(\frac{\beta}{(\sqrt{\beta}-1)^2}\big)^{\frac{\beta}{2\sqrt{\beta}-1}}$; *and in the case that* (3.6) *holds for some* $\beta \in (0,1]$, $M(\cdot\,; r(\cdot), \theta(\cdot)) \in L_{\mathcal{F}}^{\sqrt{\beta}-}(\Omega; C([0,T];\mathbb{R}))$ *and* (3.21) *becomes*: *for any* $p < \sqrt{\beta}$,

$$E\Big[\sup_{t\in[0,T]} M\big(t; r(\cdot), \theta(\cdot)\big)^p\Big] \leqslant C\big\{E\big[e^{\frac{\beta}{2}\int_0^T |\theta(s)|^2\,ds}\big]\big\}^{\frac{\sqrt{\beta}-p}{\sqrt{\beta}}}, \tag{3.23}$$

with $C = e^{\bar{a}p}\big(\frac{p^2}{(\sqrt{\beta}-p)^2}\big)^{\frac{p^2}{\sqrt{\beta}(2p-\sqrt{\beta})}}$.

Proof. (i) Since (3.6) holds for $\beta > 1$, $M(\cdot\,; 0, \theta(\cdot))$ is a martingale. Now, let $p = \frac{\alpha\beta}{\beta+\alpha(2\sqrt{\beta}-1)} < \alpha$. Then $\frac{p\alpha}{\alpha-p} = \frac{\beta}{2\sqrt{\beta}-1} > 1$. Thus, we have (noting Doob's inequality and (3.10) with $\bar{a} = 0$)

$$\begin{aligned}
&E\Big[\sup_{t\in[0,T]} M\big(t; r(\cdot), \theta(\cdot)\big)^p\Big] \\
&\quad = E\Big[\sup_{t\in[0,T]} M\big(t; 0, \theta(\cdot)\big)^p e^{-p\int_0^t r(s)\,ds}\Big] \\
&\quad \leqslant \Big\{E\Big[\sup_{t\in[0,T]} M\big(t; 0, \theta(\cdot)\big)^{\frac{p\alpha}{\alpha-p}}\Big]\Big\}^{\frac{\alpha-p}{\alpha}} \Big\{E\Big[\sup_{t\in[0,T]} e^{-\alpha\int_0^t r(s)\,ds}\Big]\Big\}^{\frac{p}{\alpha}} \\
&\quad \leqslant \Big(\frac{p\alpha}{p\alpha-\alpha+p}\Big)^p \big\{E\big[M\big(T; 0, \theta(\cdot)\big)^{\frac{p\alpha}{\alpha-p}}\big]\big\}^{\frac{\alpha-p}{\alpha}} \Big\{E\Big[\sup_{t\in[0,T]} e^{-\alpha\int_0^t r(s)\,ds}\Big]\Big\}^{\frac{p}{\alpha}} \\
&\qquad \cdot \Big\{E\Big[\sup_{t\in[0,T]} e^{-\alpha\int_0^t r(s)\,ds}\Big]\Big\}^{\frac{\beta}{\beta+\alpha(2\sqrt{\beta}-1)}} \\
&\quad \leqslant C\big\{E\big[e^{\frac{\beta}{2}\int_0^T |\theta(s)|^2\,ds}\big]\big\}^{\frac{\alpha(\sqrt{\beta}-1)}{\beta+\alpha(2\sqrt{\beta}-1)}} \Big\{E\Big[\sup_{t\in[0,T]} e^{-\alpha\int_0^t r(s)\,ds}\Big]\Big\}^{\frac{\beta}{\beta+\alpha(2\sqrt{\beta}-1)}}.
\end{aligned}$$

This proves (3.20).

(ii) For any $p \in \left(\frac{\alpha\sqrt{\beta}}{2\alpha+\sqrt{\beta}}, \frac{\alpha\sqrt{\beta}}{\alpha+\sqrt{\beta}}\right)$, let

$$\gamma = \frac{\sqrt{\beta}[(2\alpha+\sqrt{\beta})p - \alpha\sqrt{\beta}]}{\alpha p^2} > 0.$$

Then using $p < \frac{\alpha\sqrt{\beta}}{\alpha+\sqrt{\beta}}$, we have that

$$0 < p\gamma = \frac{\sqrt{\beta}[(2\alpha+\sqrt{\beta})p - \alpha\sqrt{\beta}]}{\alpha p} < \sqrt{\beta}.$$

On the other hand, since $p < \frac{\alpha\sqrt{\beta}}{\alpha+\sqrt{\beta}} < \alpha$, we have

$$\left[\alpha\sqrt{\beta} - (\alpha+\sqrt{\beta})p\right]^2 > 0,$$

which is equivalent to

$$\gamma < \frac{\alpha}{\alpha - p}. \tag{3.24}$$

Let us denote $\beta' = \frac{\beta}{p^2\gamma^2} > 1$. Then

$$E\left[e^{\frac{\beta'}{2}\int_0^T |p\gamma\theta(s)|^2\,ds}\right] = E\left[e^{\frac{\beta}{2}\int_0^T |\theta(s)|^2\,ds}\right] < \infty. \tag{3.25}$$

This implies that $M(\cdot\,;0, p\gamma\theta(\cdot))$ is a martingale, and to which (3.14) is applicable. Thus, by Doob's inequality, we have (since $p\gamma < \sqrt{\beta} \leqslant 1$, one has $p\gamma - 1 < 0$; also note (3.24))

$$\begin{aligned}
&E\Big[\sup_{t\in[0,T]} M\big(t; r(\cdot), \theta(\cdot)\big)^p\Big] \\
&\quad\leqslant E\Big[\sup_{t\in[0,T]} M\big(t; 0, p\gamma\theta(\cdot)\big)^{\frac{1}{\gamma}} e^{-p\int_0^t r(s)\,ds}\Big] \\
&\quad\leqslant \Big\{E\Big[\sup_{t\in[0,T]} M\big(t; 0, p\gamma\theta(\cdot)\big)^{\frac{\alpha}{\gamma(\alpha-p)}}\Big]\Big\}^{\frac{\alpha-p}{\alpha}} \Big\{E\Big[\sup_{t\in[0,T]} e^{-\alpha\int_0^t r(s)\,ds}\Big]\Big\}^{\frac{p}{\alpha}} \\
&\quad\leqslant \left(\frac{\alpha}{\alpha-\gamma(\alpha-p)}\right)^{\frac{1}{\gamma}} \Big\{E\Big[M\big(T; 0, p\gamma\theta(\cdot)\big)^{\frac{\alpha}{\gamma(\alpha-p)}}\Big]\Big\}^{\frac{\alpha-p}{\alpha}} \\
&\qquad\cdot \Big\{E\Big[\sup_{t\in[0,T]} e^{-\alpha\int_0^t r(s)\,ds}\Big]\Big\}^{\frac{p}{\alpha}} \Big\{E\Big[\sup_{t\in[0,T]} e^{-\alpha\int_0^t r(s)\,ds}\Big]\Big\}^{\frac{p}{\alpha}} \\
&\quad\leqslant C\Big\{E\Big[e^{\frac{\beta}{2}\int_0^T |\theta(s)|^2\,ds}\Big]\Big\}^{\frac{\alpha\sqrt{\beta}-(\alpha+\sqrt{\beta})p}{\alpha\sqrt{\beta}}} \Big\{E\Big[\sup_{t\in[0,T]} e^{-\alpha\int_0^t r(s)\,ds}\Big]\Big\}^{\frac{p}{\alpha}}
\end{aligned} \tag{3.26}$$

proving (3.21).

(iii) Let (3.8) holds for some $\bar{a} \in \mathbb{R}$. Then when $\beta > 1$, $M(\cdot\,;0, \theta(\cdot))$ is a martingale. We define $p = \frac{\beta}{2\sqrt{\beta}-1} > 1$. By Doob's inequality, we have

$$\begin{aligned}
E\Big[\sup_{t\in[0,T]} M\big(t; r(\cdot), \theta(\cdot)\big)^p\Big] &\leqslant e^{\bar{a}p} E\Big[\sup_{t\in[0,T]} M\big(t; 0, \theta(\cdot)\big)^p\Big] \\
&\leqslant e^{\bar{a}p}\left(\frac{p}{p-1}\right)^p E\big[M\big(T; 0, \theta(\cdot)\big)^p\big] \\
&\leqslant C\Big\{E\Big[e^{\frac{\beta}{2}\int_0^T |\theta(s)|^2\,ds}\Big]\Big\}^{\frac{\sqrt{\beta}-1}{2\sqrt{\beta}-1}}.
\end{aligned}$$

This proves (3.22).

Finally, let (3.6) holds only for some $\beta \in (0,1]$. Then for any $p \in \big(\frac{\sqrt{\beta}}{2}, \sqrt{\beta}\big)$, we have

$$0 < \gamma \triangleq \frac{\sqrt{\beta}(2p-\sqrt{\beta})}{p^2} < 1.$$

Further,

$$0 < p\gamma = \frac{\sqrt{\beta}(2p-\sqrt{\beta})}{p} < \sqrt{\beta} \leqslant 1.$$

Again, let us denote $\beta' = \frac{\beta}{p^2\gamma^2} > 1$. Then (3.25) holds and $M(\cdot\,;0,p\gamma\theta(\cdot))$ is a martingale. Thus, by Doob's inequality, we have (similar to (3.26))

$$\begin{aligned} E\Big[\sup_{t\in[0,T]} M\big(t;r(\cdot),\theta(\cdot)\big)^p\Big] &\leqslant e^{\bar{a}p} E\Big[\sup_{t\in[0,T]} M\big(t;0,p\gamma\theta(\cdot)\big)^{\frac{1}{\gamma}}\Big] \\ &\leqslant \frac{e^{\bar{a}p}}{(1-\gamma)^{\frac{1}{\gamma}}} E\big[M\big(T;0,p\gamma\theta(\cdot)\big)^{\frac{1}{\gamma}}\big] \\ &\leqslant C\big\{E\big[e^{\frac{\beta}{2}\int_0^T|\theta(s)|^2\,ds}\big]\big\}^{\frac{\sqrt{\beta}-p}{\sqrt{\beta}}}. \end{aligned}$$

Hence, (3.23) follows. □

Similar to the remarks we made right after Theorem 3.2, a sufficient condition for

$$M\big(\cdot\,;r(\cdot),\theta(\cdot)\big) \in L^{1+}_{\mathcal{F}}\big(\Omega;C\big([0,T];\mathbb{R}\big)\big),$$

is $\beta > 1$ and $\alpha > \frac{\beta}{(\sqrt{\beta}-1)^2}$.

Next, by (3.1), we have

$$M\big(t;r(\cdot),\theta(\cdot)\big)^{-1} = M\big(t;-\big|\theta(\cdot)\big|^2 - r(\cdot), -\theta(\cdot)\big). \tag{3.27}$$

Thus, Theorem 3.4 leads to some estimates for $M(\cdot\,;r(\cdot),\theta(\cdot))^{-1}$, which will play an interesting role in estimating adapted solutions.

Theorem 3.5. *Let* (3.6) *holds for some* $\beta > 1$*, and*

$$E\Big[\sup_{t\in[0,T]} e^{\alpha_0\int_0^t r(s)\,ds}\Big] < \infty, \tag{3.28}$$

for some $\alpha_0 > 0$*. Then* $M(\cdot\,;r(\cdot),\theta(\cdot))^{-1} \in L_{\mathcal{F}}^{\frac{\alpha_0\beta}{\beta+\alpha_0(2\sqrt{\beta}+1)}}(\Omega;C([0,T];\mathbb{R}))$ *and*

$$\begin{aligned} &E\Big[\sup_{t\in[0,T]} M\big(t;r(\cdot),\theta(\cdot)\big)^{-\frac{\alpha_0\beta}{\beta+\alpha_0(2\sqrt{\beta}+1)}}\Big] \\ &\quad\leqslant C\big\{E\big[e^{\frac{\beta}{2}\int_0^T|\theta(s)|^2\,ds}\big]\big\}^{\frac{\alpha_0(\sqrt{\beta}+1)}{\beta+\alpha_0(2\sqrt{\beta}+1)}} \Big\{E\Big[\sup_{t\in[0,T]} e^{\alpha_0\int_0^t r(s)\,ds}\Big]\Big\}^{\frac{\beta}{\beta+\alpha_0(2\sqrt{\beta}+1)}}, \end{aligned} \tag{3.29}$$

with C *depending on* α_0 *and* β.

(ii) *Suppose* (3.6) *and* (3.28) *hold for some* $\beta \in (0,1]$ *and* $\alpha_0 > 0$*, respectively. Then*

$$M\big(\cdot\,; r(\cdot), \theta(\cdot)\big)^{-1} \in L_{\mathcal{F}}^{\frac{\alpha_0\beta}{\beta+\alpha_0(\sqrt{\beta}+2)}-}\big(\Omega; C\big([0,T];\mathbb{R}\big)\big),$$

and the following estimate holds for any $p \in \big(\frac{\alpha_0\beta}{\beta+2\alpha_0(\sqrt{\beta}+1)}, \frac{\alpha_0\beta}{\beta+\alpha_0(\sqrt{\beta}+2)}\big)$:

$$\begin{aligned} &E\Big[\sup_{t\in[0,T]} M\big(t; r(\cdot), \theta(\cdot)\big)^{-p}\Big] \\ &\quad\leqslant C\big\{E\big[e^{\frac{\beta}{2}\int_0^T |\theta(s)|^2\,ds}\big]\big\}^{1+\frac{2p}{\beta}-\frac{\alpha_0\sqrt{\beta}p}{\alpha_0\beta-p(2\alpha_0+\beta)}}\Big\{E\Big[\sup_{t\in[0,T]} e^{-\alpha_0\int_0^t r(s)\,ds}\Big]\Big\}^{\frac{p}{\alpha_0}}, \end{aligned} \tag{3.30}$$

with C *depending on* α_0, β, *and* p.

(iii) *Let the following hold for some* $\bar{a}_0 \in \mathbb{R}$:

$$\sup_{t\in[0,T]} \int_0^t r(s)\,ds \leqslant \bar{a}_0, \quad \text{a.s.} \tag{3.31}$$

Then, in the case (3.6) *holds for some* $\beta > 1$, *we have* $M(\cdot\,; a(\cdot), \theta(\cdot))^{-1} \in L_{\mathcal{F}}^{\frac{\beta}{2\sqrt{\beta}+1}}(\Omega; C([0,T];\mathbb{R}))$ *and* (3.29) *becomes*

$$E\Big[\sup_{t\in[0,T]} M\big(t; r(\cdot), \theta(\cdot)\big)^{-\frac{\beta}{2\sqrt{\beta}+1}}\Big] \leqslant C\big\{E\big[e^{\frac{\beta}{2}\int_0^T |\theta(s)|^2\,ds}\big]\big\}^{\frac{\sqrt{\beta}+1}{2\sqrt{\beta}+1}}, \tag{3.32}$$

with C *depending on* $\bar{a}_0$ *and* β; *and in the case that* (3.6) *hods for some* $\beta \in (0,1]$, $M(\cdot\,; a(\cdot), \theta(\cdot))^{-1} \in L_{\mathcal{F}}^{\frac{\beta}{\sqrt{\beta}+2}-}(\Omega; C([0,T];\mathbb{R}))$. *Moreover,* (3.23) *becomes the following*: *for any* $p \in \big(\frac{\beta}{2(\sqrt{\beta}+1)}, \frac{\beta}{\sqrt{\beta}+2}\big)$:

$$E\Big[\sup_{t\in[0,T]} M\big(t; r(\cdot), \theta(\cdot)\big)^{-p}\Big] \leqslant C\big\{E\big[e^{\frac{\beta}{2}\int_0^T |\theta(s)|^2\,ds}\big]\big\}^{1+\frac{2p}{\beta}-\frac{\sqrt{\beta}p}{\beta-2p}}, \tag{3.33}$$

with C *depending on* $\bar{a}_0$, β, *and* p.

Proof. Note that

$$\begin{aligned} &E\Big[\sup_{t\in[0,T]} e^{\frac{\alpha_0\beta}{2\alpha_0+\beta}\int_0^t [|\theta(s)|^2+r(s)]\,ds}\Big] \\ &\quad\leqslant \big\{E\big[e^{\frac{\beta}{2}\int_0^T |\theta(s)|^2\,ds}\big]\big\}^{\frac{2\alpha_0}{2\alpha_0+\beta}}\Big\{E\Big[\sup_{t\in[0,T]} e^{\alpha_0\int_0^t r(s)\,ds}\Big]\Big\}^{\frac{\beta}{2\alpha_0+\beta}}. \end{aligned}$$

Then, due to (3.27), we can apply Theorem 3.4 to the current case with $r(\cdot)$ replaced by $-|\theta(\cdot)|^2 - r(\cdot)$ and $\alpha = \frac{\alpha_0\beta}{2\alpha_0+\beta}$. Thus, when $\beta > 1$, one has

$$
\begin{aligned}
&E\Big[\sup_{t\in[0,T]} M\big(t;r(\cdot),\theta(\cdot)\big)^{-\frac{\alpha_0\beta}{\beta+\alpha_0(2\sqrt{\beta}+1)}}\Big]\\
&\quad\leqslant C\Big\{E\Big[e^{\frac{\beta}{2}\int_0^T|\theta(s)|^2\,ds}\Big]\Big\}^{\frac{\alpha(\sqrt{\beta}-1)}{\beta+\alpha(2\sqrt{\beta}-1)}}\\
&\qquad\cdot\Big\{E\Big[\sup_{t\in[0,T]} e^{\frac{\alpha_0\beta}{2\alpha_0+\beta}\int_0^t[|\theta(s)|^2+r(s)]\,ds}\Big]\Big\}^{\frac{\beta}{\beta+\alpha(2\sqrt{\beta}-1)}}\\
&\quad\leqslant C\Big\{E\Big[e^{\frac{\beta}{2}\int_0^T|\theta(s)|^2\,ds}\Big]\Big\}^{\frac{\alpha_0(\sqrt{\beta}+1)}{\beta+\alpha_0(2\sqrt{\beta}+1)}}\Big\{E\Big[\sup_{t\in[0,T]} e^{-\alpha_0\int_0^t a(s)\,ds}\Big]\Big\}^{\frac{\beta}{\beta+\alpha_0(2\sqrt{\beta}+1)}},
\end{aligned}
$$

which proves (3.29).

(ii) Next, let $\beta\in(0,1]$. By (ii) of Theorem 3.4, for any p satisfying

$$
p\in\left(\frac{\alpha_0\beta}{\beta+2\alpha_0(\sqrt{\beta}+1)},\frac{\alpha_0\beta}{\beta+\alpha_0(\sqrt{\beta}+2)}\right)=\left(\frac{\alpha\sqrt{\beta}}{2\alpha+\sqrt{\beta}},\frac{\alpha\sqrt{\beta}}{\alpha+\sqrt{\beta}}\right)
$$

(with $\alpha=\frac{\alpha_0\beta}{2\alpha_0+\beta}$) we have

$$
\begin{aligned}
&E\Big[\sup_{t\in[0,T]} M\big(t;r(\cdot),\theta(\cdot)\big)^{-p}\Big]\\
&\quad\leqslant C\Big\{E\Big[e^{\frac{\beta}{2}\int_0^T|\theta(s)|^2\,ds}\Big]\Big\}^{1-\frac{\alpha p}{(\alpha-p)\sqrt{\beta}}}\Big\{E\Big[\sup_{t\in[0,T]} e^{\frac{\alpha_0\beta}{2\alpha_0+\beta}\int_0^t[|\theta(s)|^2+r(s)]\,ds}\Big]\Big\}^{\frac{p}{\alpha}}\\
&\quad\leqslant C\Big\{E\Big[e^{\frac{\beta}{2}\int_0^T|\theta(s)|^2\,ds}\Big]\Big\}^{1+\frac{2p}{\beta}-\frac{\alpha_0\sqrt{\beta}p}{\alpha_0\beta-p(2\alpha_0+\beta)}}\Big\{E\Big[\sup_{t\in[0,T]} e^{\alpha_0\int_0^t r(s)\,ds}\Big]\Big\}^{\frac{p}{\alpha_0}}. \qquad (3.34)
\end{aligned}
$$

This proves (3.30).

(iii) Let (3.31) hold for some $\bar a_0\in\mathbb{R}$. If (3.6) holds with $\beta>1$, then

$$
E\Big[\sup_{t\in[0,T]} e^{\frac{\beta}{2}\int_0^t[|\theta(s)|^2+r(s)]\,ds}\Big]\leqslant e^{\frac{\bar a_0\beta}{2}}E\Big[e^{\frac{\beta}{2}\int_0^T|\theta(s)|^2\,ds}\Big].
$$

Hence, due to (3.27), we can apply Theorem 3.4 to the current case with $r(\cdot)$ replaced by $-|\theta(\cdot)|^2-r(\cdot)$ and $\alpha=\frac{\beta}{2}$. Thus, when $\beta>1$, similar to (3.34), one has

$$
\begin{aligned}
&E\Big[\sup_{t\in[0,T]} M\big(t;r(\cdot),\theta(\cdot)\big)^{-\frac{\beta}{2\sqrt{\beta}+1}}\Big]\\
&\quad\leqslant C\Big\{E\Big[e^{\frac{\beta}{2}\int_0^T|\theta(s)|^2\,ds}\Big]\Big\}^{\frac{\alpha(\sqrt{\beta}-1)}{\beta+\alpha(2\sqrt{\beta}-1)}}\Big\{E\Big[\sup_{t\in[0,T]} e^{\frac{\beta}{2}\int_0^t[|\theta(s)|^2+r(s)]\,ds}\Big]\Big\}^{\frac{\beta}{\beta+\alpha(2\sqrt{\beta}-1)}}\\
&\quad\leqslant C\Big\{E\Big[e^{\frac{\beta}{2}\int_0^T|\theta(s)|^2\,ds}\Big]\Big\}^{\frac{\sqrt{\beta}+1}{2\sqrt{\beta}+1}},
\end{aligned}
$$

which gives (3.32).

In the case that $\beta\in(0,1]$, by (ii) of Theorem 3.4, for any p with (note $\alpha=\beta/2$)

$$
p\in\left(\frac{\beta}{2(\sqrt{\beta}+1)},\frac{\beta}{\sqrt{\beta}+2}\right)=\left(\frac{\alpha\sqrt{\beta}}{2\alpha+\sqrt{\beta}},\frac{\alpha\sqrt{\beta}}{\alpha+\sqrt{\beta}}\right),
$$

we have

$$
\begin{aligned}
&E\Big[\sup_{t\in[0,T]} M\big(t;r(\cdot),\theta(\cdot)\big)^{-p}\Big]\\
&\quad\leqslant C\big\{E\big[e^{\frac{\beta}{2}\int_0^T|\theta(s)|^2\,ds}\big]\big\}^{1-\frac{\alpha p}{(\alpha-p)\sqrt{\beta}}}\Big\{E\Big[\sup_{t\in[0,T]} e^{\frac{\beta}{2}\int_0^t[|\theta(s)|^2+r(s)]\,ds}\Big]\Big\}^{\frac{p}{\alpha}}\\
&\quad\leqslant C\big\{E\big[e^{\frac{\beta}{2}\int_0^T|\theta(s)|^2\,ds}\big]\big\}^{1+\frac{2p}{\beta}-\frac{\sqrt{\beta}p}{\beta-2p}}.
\end{aligned}
$$

This proves (3.33). $\square$

4. Solvability of linear BSDEs and completeness of markets

In this section, we will establish the solvability of BSDE (1.5), under different conditions on the coefficients and the terminal condition. Then the completeness of the markets will follow.

First of all, we recall the following simple consequence of Hölder's inequality: Suppose $p_1,p_2,p_3>0$ and $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}$. Then for any real functions $f_1\in L^{p_1}$ and $f_2\in L^{p_2}$, we have $f_1f_2\in L^{p_3}$. Based on this observation, we are at the position of stating and proving the first main result of this section.

Theorem 4.1. *Let* (3.4), (3.6) *and* (3.28) *hold with* $\beta,\alpha,\alpha_0>0$ *satisfy the following*:

$$
\begin{cases}\beta>\max\big\{3+2\sqrt{2},\big(\frac{4\alpha\alpha_0}{\alpha\alpha_0-\alpha-\alpha_0}\big)^2\big\}, & \alpha>\frac{\beta}{(\sqrt{\beta}-1)^2},\\ \alpha_0>\frac{\beta}{(\sqrt{\beta}-1)^2-2}, & \alpha\alpha_0>\alpha+\alpha_0.\end{cases}\tag{4.1}
$$

Then for any $\xi\in L^{p+}_{\mathcal{F}_T}(\Omega;\mathbb{R})$ *with*

$$
p=\frac{\alpha\alpha_0\sqrt{\beta}}{\sqrt{\beta}(\alpha\alpha_0-\alpha-\alpha_0)-4\alpha\alpha_0}>1,\tag{4.2}
$$

BSDE (1.5) *admits a unique adapted solution* $(Y(\cdot),Z(\cdot))$ *with*

$$
Y(\cdot)\in L^{1+}_{\mathcal{F}}\big(\Omega;C\big([0,T];\mathbb{R}\big)\big),\qquad Z(\cdot)\in L^{1+}_{\mathcal{F}}\big(\Omega;L^2\big(0,T;\mathbb{R}^d\big)\big).
$$

Proof. In what follows, we denote $M(\cdot)=M(\cdot\,;r(\cdot),\theta(\cdot))$. First of all, from Theorem 3.2, we know that under (4.1), the following holds:

$$
M(T)\in L^{p_1}_{\mathcal{F}_T}\big(\Omega;C\big([0,T];\mathbb{R}\big)\big),\tag{4.3}
$$

with

$$
p_1=\frac{\alpha\beta}{\beta+\alpha(2\sqrt{\beta}-1)}>1.\tag{4.4}
$$

On the other hand, for any $\xi\in L^{p_2}_{\mathcal{F}_T}(\Omega)$, $p_2>p$ with $p>1$ given by (4.2), we have

$$
M(T)\xi\in L^{p_3}_{\mathcal{F}_T}(\Omega;\mathbb{R}),\tag{4.5}
$$

with

$$p_3=\frac{p_1p_2}{p_1+p_2}=\frac{\alpha\beta p_2}{\alpha\beta+p_2[\beta+\alpha(2\sqrt{\beta}-1)]}$$
$$>\frac{\alpha\beta\frac{\alpha\alpha_0\sqrt{\beta}}{\sqrt{\beta}(\alpha\alpha_0-\alpha-\alpha_0)-4\alpha\alpha_0}}{\alpha\beta+\frac{\alpha\alpha_0\sqrt{\beta}}{\sqrt{\beta}(\alpha\alpha_0-\alpha-\alpha_0)-4\alpha\alpha_0}[\beta+\alpha(2\sqrt{\beta}-1)]}$$
$$=\frac{\alpha_0\beta}{\alpha_0[(\sqrt{\beta}-1)^2-2]-\beta}>1.$$

Consequently, by [3], we know that (1.13) admits a unique adapted solution

$$\big(\widetilde{Y}(\cdot),\widetilde{Z}(\cdot)\big)\in L^{p_3}_{\mathcal{F}}\big(\Omega;C\big([0,T];\mathbb{R}\big)\big)\times L^{p_3}_{\mathcal{F}}\big(\Omega;L^2\big(0,T;\mathbb{R}^d\big)\big),$$

and the following estimate holds:

$$E\left\{\sup_{t\in[0,T]}\big|\widetilde{Y}(t)\big|^{p_3}+\left[\int_0^T\big|\widetilde{Z}(t)\big|^2\,dt\right]^{\frac{p_3}{2}}\right\}\leqslant C\big\{E\big[M(T)^{p_1}\big]\big\}^{\frac{p_3}{p_1}}\big\{E\big[|\xi|^{p_2}\big]\big\}^{\frac{p_3}{p_2}}. \quad (4.6)$$

Next, by (4.1) we know that

$$M(\cdot)^{-1}\in L^{p_4}_{\mathcal{F}}\big(\Omega;C\big([0,T];\mathbb{R}\big)\big), \quad (4.7)$$

with

$$p_4=\frac{\alpha_0\beta}{\beta+\alpha_0(2\sqrt{\beta}+1)}>1.$$

Now, we define $(Y(\cdot),Z(\cdot))$ by (1.14). Then

$$E\Big[\sup_{t\in[0,T]}\big|Y(t)\big|^{p_5}\Big]$$
$$\leqslant\Big\{E\Big[\sup_{t\in[0,T]}M(t)^{-p_4}\Big]\Big\}^{\frac{p_5}{p_4}}\Big\{E\Big[\sup_{t\in[0,T]}\big|\widetilde{Y}(t)\big|^{p_3}\Big]\Big\}^{\frac{p_5}{p_3}}$$
$$\leqslant C\Big\{E\Big[\sup_{t\in[0,T]}M(t)^{-p_4}\Big]\Big\}^{\frac{p_5}{p_4}}\big\{E\big[M(T)^{p_1}\big]\big\}^{\frac{p_5}{p_1}}\big\{E|\xi|^{p_2}\big\}^{\frac{p_5}{p_2}}\leqslant C\big\{E|\xi|^{p_2}\big\}^{\frac{p_5}{p_2}} \quad (4.8)$$

with

$$\frac{1}{p_5}=\frac{1}{p_3}+\frac{1}{p_4}<\frac{\alpha_0[(\sqrt{\beta}-1)^2-2]+\alpha_0(2\sqrt{\beta}+1)}{\alpha_0\beta}=1. \quad (4.9)$$

On the other hand, we note that for any $p\geqslant 1$,

$$E\left[\int_0^T|\theta(t)|^2\,dt\right]^p\leqslant E\big[e^{\frac{\beta}{2}\int_0^T|\theta(t)|^2\,dt}\big]. \quad (4.10)$$

Hence, by taking $\varepsilon\in(0,p_5)$, one has

$$
\begin{aligned}
&E\Bigg[\int_0^T \big|Z(t)\big|^2\,dt\Bigg]^{\frac{p_5-\varepsilon}{2}} \\
&\quad = E\Bigg[\int_0^T \big|M(t)^{-1}\big[\widetilde Z(t)+\widetilde Y(t)\theta(t)\big]\big|^2\,dt\Bigg]^{\frac{p_5-\varepsilon}{2}} \\
&\quad \leqslant C\Bigg\{\Big[E\Big(\sup_{t\in[0,T]} M(t)^{-p_4}\Big)\Big]^{\frac{p_5-\varepsilon}{p_4}}\Bigg[\bigg(E\int_0^T \big|\widetilde Z(t)\big|^2\,dt\bigg)^{\frac{p_3}{2}}\Bigg]^{\frac{p_5-\varepsilon}{p_3}} \\
&\qquad + \Big[E\Big(\sup_{t\in[0,T]} M(t)^{-p_4}\Big)\Big]^{\frac{p_5-\varepsilon}{p_4}}\Big[E\Big(\sup_{t\in[0,T]} \widetilde Y(t)^{p_3}\Big)\Big]^{\frac{p_5-\varepsilon}{p_3}} \\
&\qquad \cdot\Bigg[E\bigg(\int_0^T \big|\theta(t)\big|^2\,dt\bigg)^{\frac{p_5(p_5-\varepsilon)}{2\varepsilon}}\Bigg]^{\frac{\varepsilon}{p_5}}\Bigg\} \leqslant C\big\{E|\xi|^{p_2}\big\}^{\frac{p_5-\varepsilon}{p_2}}.
\end{aligned}
\tag{4.11}
$$

Combining the above, we obtain our conclusion. □

Now, we look at the case that (4.1) does not hold. In this case, BSDE (1.5) might have no adapted solutions $(Y(\cdot), Z(\cdot))$ in $L^{1+}_{\mathcal F}(\Omega; C([0,T];\mathbb R) \times L^{1+}_{\mathcal F}(\Omega; L^2(0,T;\mathbb R^d))$. But one can get adapted solutions with less integrability. We now present the following result.

Theorem 4.2. *Let* (3.4)*,* (3.6)*, and* (3.28) *hold with* $\beta, \alpha, \alpha_0 > 0$ *satisfying*:

$$
\beta > 1, \qquad \alpha > \frac{\beta}{(\sqrt{\beta}-1)^2}. \tag{4.12}
$$

Then for any $\xi \in L^{p+}_{\mathcal F_T}(\Omega;\mathbb R)$ *with*

$$
p = \frac{\alpha\beta}{\alpha(\sqrt{\beta}-1)^2-\beta} > 1, \tag{4.13}
$$

BSDE (1.5) *admits a unique adapted solution* $(Y(\cdot), Z(\cdot))$ *such that*

$$
Y(\cdot) \in L^{q+}_{\mathcal F}\big(\Omega; C\big([0,T];\mathbb R^n\big)\big), \qquad Z(\cdot) \in L^{q+}_{\mathcal F}\big(\Omega; L^2\big(0,T;\mathbb R^d\big)\big), \tag{4.14}
$$

with

$$
q = \frac{\alpha_0\beta}{\alpha_0(\sqrt{\beta}+1)^2+\beta} \in (0,1). \tag{4.15}
$$

Proof. In the current case, we still have (4.3)–(4.4). Then for any $\xi \in L^{p_2}_{\mathcal F_T}(\Omega;\mathbb R)$, $p_2 > p$ with $p > 1$ given by (4.13), we still have (4.5), and with

$$
\begin{aligned}
p_3 &= \frac{p_1p_2}{p_1+p_2} = \frac{\alpha\beta p_2}{\alpha\beta + p_2[\beta+\alpha(2\sqrt{\beta}-1)]} \\
&> \frac{\alpha\beta\frac{\alpha\beta}{\alpha(\sqrt{\beta}-1)^2-\beta}}{\alpha\beta+\frac{\alpha\beta}{\alpha(\sqrt{\beta}-1)^2-\beta}[\beta+\alpha(2\sqrt{\beta}-1)]} = 1.
\end{aligned}
$$

Thus, we still have (4.6). Next, we define $(Y(\cdot), Z(\cdot))$ by (1.14), which is an adapted solution to BSDE (1.5). Now, by Theorem 3.5, we have (4.7), but with $p_4 > 0$ is not necessarily larger than 1. Then we only have

$$\frac{1}{p_5} = \frac{1}{p_3} + \frac{1}{p_4} < 1 + \frac{\beta + \alpha_0(2\sqrt{\beta}+1)}{\alpha_0\beta} = \frac{\alpha_0(\sqrt{\beta}+1)^2 + \beta}{\alpha_0\beta}.$$

Then similar to (4.8)–(4.11), we obtain (4.14)–(4.15). $\square$

The above cases are not too bad. But, if (4.12) fails, then $M(T)$ is not necessarily in $L^1_{\mathcal{F}_T}(\Omega;\mathbb{R})$. Thus, we ask: Does BSDE (1.13) admit an adapted solution $(\widetilde{Y}(\cdot), \widetilde{Z}(\cdot))$ if $\tilde{\xi}$ is not even in $L^1_{\mathcal{F}_T}(\Omega;\mathbb{R})$? Surprisingly, we do have a positive answer to this question due to the results of Dudley [2] and Garling [4]. More precisely, we have the following:

Lemma 4.3. *For any $\tilde{\xi} \in L^p_{\mathcal{F}_T}(\Omega;\mathbb{R})$ with $0 < p < 1$, BSDE* (1.13) *admits a unique adapted solution $(\widetilde{Y}(\cdot), \widetilde{Z}(\cdot)) \in L^p_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R})) \times L^p_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{R}^d))$.*

Proof. By [4], together with Burkholder–Davis–Gundy inequality [5], for any $\tilde{\xi} \in L^p_{\mathcal{F}_T}(\Omega;\mathbb{R})$, one can find a unique $\widetilde{Z}(\cdot) \in L^p_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{R}^d))$ such that

$$\tilde{\xi} = \int_0^T \langle \widetilde{Z}(s), dW(s) \rangle.$$

Then by defining

$$\widetilde{Y}(t) = \int_0^t \langle \widetilde{Z}(s), dW(s) \rangle, \quad t \in [0,T],$$

we see that $(\widetilde{Y}(\cdot), \widetilde{Z}(\cdot)) \in L^p_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R})) \times L^p_{\mathcal{F}}(\Omega; L^2(0,T;\mathbb{R}^d))$ is the unique adapted solution to (1.13). $\square$

We now can prove the following result which can take care of the case that (4.12) fails.

Theorem 4.4. *Let* (3.4), (3.6), *and* (3.28) *hold for some $\beta, \alpha, \alpha_0 > 0$. Then for any $\xi \in L^p_{\mathcal{F}_T}(\Omega;\mathbb{R})$ with $p > 0$, BSDE* (1.5) *admits a unique adapted solution*

$$\big(Y(\cdot), Z(\cdot)\big) \in L^q_{\mathcal{F}}\big(\Omega; C\big([0,T];\mathbb{R}\big)\big) \times L^{q-}_{\mathcal{F}}\big(\Omega; L^2\big(0,T;\mathbb{R}^d\big)\big),$$

with

$$q = \frac{\alpha\alpha_0\beta p}{\alpha\alpha_0\beta + \alpha_0\beta p + \alpha\beta p + \alpha\alpha_0 p + \alpha\alpha_0 p\sqrt{\beta}(2+\sqrt{\beta})} \in (0,1). \tag{4.16}$$

Proof. By Theorem 3.2(i), we know that $M(T) \in L^{\frac{\alpha}{\alpha+1}}_{\mathcal{F}_T}(\Omega;\mathbb{R})$. Thus, for any $\xi \in L^p_{\mathcal{F}_T}(\Omega;\mathbb{R})$, we have

$$\tilde{\xi} \equiv M(T)\xi \in L^{p_3}_{\mathcal{F}_T}(\Omega;\mathbb{R}),$$

with

$$p_3 = \frac{\frac{\alpha}{\alpha+1} p}{\frac{\alpha}{\alpha+1} + p} = \frac{\alpha p}{\alpha + p(\alpha+1)} \in (0,1).$$

By Lemma 4.3, we know that (1.13) admits a unique adapted solution

$$\big(\widetilde{Y}(\cdot), \widetilde{Z}(\cdot)\big) \in L^{p_3}_{\mathcal{F}}\big(\Omega; C\big([0,T]; \mathbb{R}\big)\big) \times L^{p_3}_{\mathcal{F}_T}\big(\Omega; L^2\big(0,T;\mathbb{R}^d\big)\big).$$

Then define $(Y(\cdot), Z(\cdot))$ by (1.14), we have an adapted solution to BSDE (1.5). Next, by Theorem 3.5(i), we know that (4.7) holds with $p_4 = \frac{\alpha_0 \beta}{\beta + \alpha_0(2\sqrt{\beta}+1)} \in (0,1)$. Then the rest of the proof similar to that of Theorem 4.2. □

The significance of the above results is that under conditions (3.4), (3.6) and (3.28) (for some $\beta, \alpha, \alpha_0 > 0$), BSDE (1.5) admits an adapted solution with some weaker integrability which can actually be estimated in terms of α, α_0, β and p. Note that (4.16) implies

$$0 < q < \min\{\alpha, \alpha_0, \beta, p, 1\}.$$

As a byproduct, let us look at the following BSDE:

$$\begin{cases} dY(t) = [r(t)Y(t) + \langle \theta(t), Z(t)\rangle + f(t)]\,dt + \langle Z(t), \theta(t)\,dW(t)\rangle, \\ Y(T) = \xi. \end{cases} \tag{4.17}$$

If $(Y(\cdot), Z(\cdot))$ is an adapted solution of (4.17), then taking $M(\cdot)$ as (1.11), we have

$$d\big[M(t)Y(t)\big] = M(t)f(t)\,dt + M(t)\big\langle Z(t) - Y(t)\theta(t), dW(t)\big\rangle.$$

Thus, if we define (comparing with (1.12))

$$\begin{cases} \widetilde{Y}(t) = M(t)Y(t) - \int_0^t M(s)f(s)\,ds, \\ \widetilde{Z}(t) = M(t)[Z(t) - Y(t)\theta(t)], \end{cases}$$

then $(\widetilde{Y}(\cdot), \widetilde{Z}(\cdot))$ is an adapted solution to BSDE (1.13) with

$$\tilde{\xi} = M(T)\xi - \int_0^T M(s)f(s)\,ds.$$

Now, (1.14) will be replaced by

$$\begin{cases} Y(t) = M(t)^{-1}[\widetilde{Y}(t) + \int_0^t M(s)f(s)\,ds], \\ Z(t) = M(t)^{-1}[\widetilde{Z}(t) + \widetilde{Y}(t)\theta(t)]. \end{cases}$$

Hence, BSDE (4.17) can be handled similarly. We omit the details here.

Now, we discuss the completeness of the market described by (1.1), based on the solvability of BSDEs with unbounded coefficients. Let us introduce the following definition.

Definition 4.5. Let $\mathcal{H} \subseteq L^0_{\mathcal{F}_T}(\Omega; \mathbb{R})$ be a space of contingent claims and $\Pi \subseteq \Pi^0[0,T]$ be a subspace of portfolios.

(i) A contingent claim $\xi \in \mathcal{H}$ is said to be Π-replicatable if BSDE (1.2) admits an adapted solution $(Y(\cdot), \pi(\cdot))$ with some $\pi(\cdot) \in \Pi$.

(ii) Market is said to be $\{\Pi, \mathcal{H}\}$-complete if any contingent claim $\xi \in \mathcal{H}$ is replicatable by a portfolio $\pi(\cdot) \in \Pi$.

In the above, $\mathcal{H}$ could be $L^p_{\mathcal{F}_T}(\Omega; \mathbb{R})$, and Π could be $\Pi^p[0,T]$ or $\Pi^{p\pm}[0,T]$. According to Theorems 4.1, 4.2, and 4.4, we have the following result (note (1.7), and the definitions of $\Pi^p[0,T]$ and $\Pi^{p\pm}[0,T]$).

Theorem 4.6. *Let* (1.3) *hold, and* $\theta(\cdot)$ *be defined by* (1.4)*. Let* (3.4)*,* (3.6)*, and* (3.28) *hold with some* $\beta, \alpha, \alpha_0 > 0$*. Then the following are true*:

(i) *For any* $p > 0$*, the market is* $\{\Pi^{q-}[0,T], L^p_{\mathcal{F}_T}(\Omega; \mathbb{R})\}$*-complete with* q *given by* (4.16);
(ii) *If* (4.12) *holds, the market is* $\{\Pi^{q+}[0,T], L^{p+}_{\mathcal{F}_T}(\Omega; \mathbb{R})\}$*-complete with* p *and* q *given by* (4.13) *and* (4.15)*, respectively*;
(iii) *If* (4.1) *holds, the market is* $\{\Pi^{1+}[0,T], L^{p+}_{\mathcal{F}_T}(\Omega; \mathbb{R})\}$*-complete with* p *given by* (4.2).

The above result shows that although the process $\theta(\cdot)$ and/or the interest rate process $r(\cdot)$ might be unbounded, the market will still be complete in some sense. To conclude this section, let us point out that by [8], one has the following result.

Proposition 4.7. *Let*

$$r(\cdot) \in L^\infty_{\mathcal{F}}\big(\Omega; L^1(0,T;\mathbb{R})\big), \qquad \theta(\cdot) \in L^\infty_{\mathcal{F}}\big(\Omega; L^2\big(0,T;\mathbb{R}^d\big)\big). \tag{4.18}$$

Then for any $\xi \in L^\infty_{\mathcal{F}_T}(\Omega; \mathbb{R})$*, BSDE* (1.5) *admits a unique adapted solution*

$$\big(Y(\cdot), Z(\cdot)\big) \in L^\infty_{\mathcal{F}}\big(\Omega; C\big([0,T];\mathbb{R}\big)\big) \times L^2_{\mathcal{F}}\big(0,T;\mathbb{R}^d\big).$$

One sees that under (4.18), conditions (3.4), (3.6), and (3.28) hold for any $\beta, \alpha, \alpha_0 > 0$. By Proposition 4.7, we have that the market is $\{\Pi^{2-}[0,T], L^\infty_{\mathcal{F}_T}(\Omega; \mathbb{R})\}$-complete.

5. Some examples and remarks

Let us first present two illustrative examples.

Suppose the interest rate process $r(\cdot)$ follows the CIR model (with a one-dimensional Brownian motion $W(\cdot)$):

$$dr(t) = \big[a_0 - a_1 r(t)\big]\,dt + \sigma_0\sqrt{r(t)}\,dW(t), \quad r(0) > 0, \tag{5.1}$$

with $a_0, a_1, \sigma_0 > 0$ and the following holds:

$$4a_0 \leqslant \sigma_0^2. \tag{5.2}$$

Then it is standard that $r(\cdot)$ exists and non-negative. Thus, (3.4) holds for any $\alpha > 0$. Take $\alpha_0 > 0$ satisfying:

$$\alpha_0 < \frac{2a_1}{T(e^{a_1 T} - 1)\sigma_0^2}. \tag{5.3}$$

Then by some direct calculations, we obtain (3.28) (for more general discussion concerning this matter, see [17]). We consider a market in which there are one bond, with interest process $r(\cdot)$, and one stock with appreciation rate process $b(\cdot)$, and volatility process $\sigma(\cdot)$. Assume that $b(\cdot) - r(\cdot)$ is a (non-negative) bounded process and $\sigma(\cdot)$ and $\sigma(\cdot)^{-1}$ are all bounded. Then $\theta(\cdot)$ defined by (1.4) will be bounded. Hence, (3.6) holds for any $\beta > 0$. Consequently, by Theorem 4.2, the market is $\{\Pi^{1-}[0,T], L^{1+}_{\mathcal{F}_T}(\Omega;\mathbb{R})\}$-complete. In addition to (5.3), if $\alpha_0 > 1$, by Theorem 4.6(iii), the market is $\{\Pi^{1+}[0,T], L^{\frac{\alpha_0}{\alpha_0-1}+}_{\mathcal{F}_T}(\Omega;\mathbb{R})\}$-complete.

Next, we assume that $r(\cdot)$ follows Vasicek's model

$$dr(t) = \big[a_0 - a_1 r(t)\big]\,dt + \sigma_0\,dW(t), \tag{5.4}$$

with $a_0, a_1, \sigma_0 > 0$. By some direct calculation (see [17]), we have (3.4) and (3.28) for any $\alpha, \alpha_0 > 0$. Consider a market in which there are one bond and one stock. Suppose that $b(\cdot)$, $\sigma(\cdot)$, and $\sigma(\cdot)^{-1}$ are bounded. Note that $\theta(\cdot)$ defined by (1.4) is not necessarily bounded. Now, take $\beta > 0$ satisfying

$$\beta < \frac{2a_1}{T|\sigma(\cdot)^{-1}|^2_\infty \sigma_0^2 (e^{2a_1 T} - 1)}. \tag{5.5}$$

Then by a direct calculation, we have (3.6). Consequently, by Theorems 4.1, 4.2 and 4.4, we have the following conclusions:

(i) For any $p > 0$, the market is $\{\Pi^{\frac{\beta p}{p(\sqrt{\beta}+1)^2+\beta}-}[0,T], L^p_{\mathcal{F}_T}(\Omega;\mathbb{R})\}$-complete.

(ii) In addition to (5.5), if $\beta > 1$, the market is $\{\Pi^{\frac{\beta}{(\sqrt{\beta}+1)^2}-}[0,T], L^{\frac{\beta}{(\sqrt{\beta}-1)^2}+}_{\mathcal{F}_T}(\Omega;\mathbb{R})\}$-complete.

(iii) In addition to (5.5), if $\beta > 16$, the market is $\{\Pi^{1+}[0,T], L^{\frac{\sqrt{\beta}}{\sqrt{\beta}-4}+}_{\mathcal{F}_T}(\Omega;\mathbb{R})\}$-complete.

Note that the $\{\Pi^q[0,T], L^p_{\mathcal{F}_T}(\Omega;\mathbb{R})\}$-completeness of the market is harder to obtained if q is increased and/or p is decreased. Thus, in the above, as the conclusions are concerned, (iii) is stronger than (ii), and (ii) is stronger than (i).

In the above two examples, we only considered the case that $r(\cdot)$ is unbounded, which leads to BSDEs with unbounded coefficients. It is not hard to cook up situations that the unboundedness is resulted from the inverse $[\sigma(\cdot)^T\sigma(\cdot)]^{-1}$, and/or the combinations of the above-mentioned two situations. Also, it is possible to consider higher dimensional cases. We prefer not to get into detailed discussions here.

To conclude this paper, we make a couple of remarks. First of all, for Black–Scholes market model, the completeness of the market is equivalent to the solvability of BSDEs. In the case that the coefficients are not necessarily bounded, we have presented some sufficient conditions under which the BSDEs are solvable which lead to the completeness of the corresponding markets. It is important to note that the spaces of the portfolios and the contingent claims are allowed to have different integrability. Second, we note that the necessity of the conditions imposed for solvability of BSDEs has not been discussed in the paper. It is desirable to know that under what kind of general meaningful conditions on the coefficients, the BSDEs are not solvable, which lead to the incompleteness of the markets. Such kind of results will give us the necessity for the conditions that we introduced in the paper. We hope to address these problems in a forthcoming paper.

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